

The Generalized Eigenvalue Problem for Certain Unsymmetric Band Matrices

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

Submitted by G. W. Stewart

ABSTRACT

Closed expressions are given for the solution to the generalized eigenproblem for unsymmetric band matrices whose elements repeat in certain ways. An application to the discretization of differential equations is described.

1. INTRODUCTION

In this paper we shall derive explicit solutions to the generalized eigenproblem

$$Ax = \lambda Bx, \quad (1.1)$$

where A and B are real $n \times n$ band matrices whose elements *repeat* in certain ways. In Sec. 2 we present a basic lemma from which we can deduce *closed expressions* (in terms of square roots) for the eigenvalues and eigenvectors, in the case that A and B are tridiagonal. This is extended in Sec. 3 to certain types of quindagonal matrices, where the eigenvalues are shown to be given by the roots of a family of polynomial equations of degree four. The results are extended in a different direction in Sec. 4 to certain block matrices.

One important application of these ideas is to the discretization of differential equations, and this is described briefly in Sec. 5, with some examples. The expressions for the eigenvalues are valuable in that they enable the stability of the discretization to be analyzed, as well as its relationship to the original differential equation.

2. TRIDIAGONAL SYSTEMS

We consider the eigenproblem (1.1) where A and B are $n \times n$ tridiagonal matrices of the form

$$A = \begin{bmatrix} d & e & & & \\ f & d & e & & \\ & f & \ddots & \ddots & \\ & & \ddots & f & d & e \\ & & & f & d \end{bmatrix}, \quad B = \begin{bmatrix} g & h & & & \\ k & g & h & & \\ & k & \ddots & \ddots & \\ & & \ddots & k & g & h \\ & & & k & g \end{bmatrix}, \quad (2.1)$$

where d, e, \dots, k are real constants. The case when B reduces to the identity matrix ($g=1, h=k=0$) is well known and has found a number of applications in the analysis of finite difference approximations to parabolic equations (Mitchell [1]). Our primary motivation for the current work is to provide a method of analysis for more complicated systems arising from finite element methods (see Griffiths and Mitchell [2]). To derive the eigensystem in the present case we define the $n \times n$ matrix

$$C = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & c & a & b \\ & & & c & a \end{bmatrix}. \quad (2.2)$$

We then have

LEMMA. If

$$a = \sqrt{bc} \xi_k \quad (2.3)$$

where

$$\xi_k = 2 \cos \frac{\pi k}{n+1} \quad (2.4)$$

and k is an integer ($1 \leq k \leq n$), then

$$\det C = 0 \quad (2.5)$$

and

$$Cv^{(k)} = 0, \quad (2.6)$$

where

$$(\mathbf{v}^{(k)})_i = \begin{cases} (-1)^{i-1} \left(\frac{c}{b}\right)^{(i-1)/2} \sin \frac{\pi j k}{n+1}, & b \neq 0 \\ \delta_{jn}, & b = 0 \end{cases} \quad i = 1, 2, \dots, n \quad (2.7)$$

and δ_{jn} is the Kronecker delta.

Proof. The case when either b or c vanishes is trivial, and so we need consider only $bc \neq 0$. Defining the diagonal matrix

$$D = \begin{bmatrix} 1 & & & & \\ & \alpha & & & \\ & & \alpha^2 & & \\ & & & \ddots & \\ & & & & \alpha^{n-1} \end{bmatrix}$$

where $\alpha = (b/c)^{1/2}$, we find

$$DCD^{-1} = \begin{bmatrix} a & (bc)^{1/2} & & & \\ (bc)^{1/2} & a & (bc)^{1/2} & & \\ & \cdot & \cdot & \ddots & \cdot \\ & & & (bc)^{1/2} & a \end{bmatrix}.$$

For an $n \times n$ matrix X of the form

$$X = \begin{bmatrix} x & 1 & & & \\ 1 & x & 1 & & \\ & \cdot & \cdot & \ddots & \cdot \\ & & & 1 & x \end{bmatrix},$$

it is shown by Rutherford [3] that

$$\det X = \prod_{k=1}^n (x - \xi_k),$$

where ξ_k is defined by (2.4). The results (2.3) and (2.5) follow by observing that $DCD^{-1} = (bc)^{1/2}X$ when $x = a/(bc)^{1/2}$.

Denoting by X_k the matrix X evaluated at $x = \xi_k$, it follows from elementary trigonometrical relations that

$$X_k \mathbf{u}^{(k)} = 0,$$

where

$$(\mathbf{u}^{(k)})_i = (-1)^i \sin \frac{\pi j k}{n+1}.$$

Thus (2.6) is established by defining $\mathbf{v}^{(k)} = D^{-1}\mathbf{u}^{(k)}$. ■

This lemma applies directly to the generalized eigenproblem (1.1) when A and B are given by (2.1), by setting $C = A - \lambda B$. This leads to

$$a = d - \lambda g, \quad b = e - \lambda h, \quad c = f - \lambda k. \quad (2.8)$$

These expressions have to be substituted into (2.3) and the resulting family of quadratic equations solved for λ . However, the antisymmetry of the values $\xi_1, \xi_2, \dots, \xi_n$ about the point $k = m$, where m is the integer part of $\frac{1}{2}(n+1)$, enables us to solve

$$a^2 = bc\xi_k^2, \quad k = 1, 2, \dots, m \quad (2.9)$$

where only the simple root $a = 0$ is taken when $m = \frac{1}{2}(n+1)$.

From (2.8) and (2.9) we then deduce that the quadratic equation

$$(g^2 - rhk)\lambda^2 - \{2dg - r(hf + ek)\}\lambda + d^2 - ref = 0, \quad (2.10)$$

with $r = \xi_k^2$, produces two eigenvalues λ_k and λ_{n+1-k} of (1.1) for each value of k , $1 \leq k \leq m$.

The distinction between the cases when n is odd and even is that in the former, there is always a real (simple) root given by

$$\lambda_{\frac{1}{2}(n+1)} = d/g. \quad (2.11)$$

Provided this fact is acknowledged, there is no loss of generality in assuming that n is even. This we shall now do and exclude the possibility $k = \frac{1}{2}(n+1)$.

Applications of this result usually require information regarding the distribution of real and complex eigenvalues [2]. Equation (2.10) has real roots provided that

$$\frac{1}{4}r(hf - ek)^2 + (hd - eg)(kd - fg) \geq 0. \quad (2.12)$$

Further, since

$$4 > r > 0 \quad (2.13)$$

($r = \xi_k^2$), we obtain the important limiting cases

$$r = 0: \quad (hd - eg)(kd - fg) \geq 0, \quad (2.14)$$

$$r = 4: \quad (hf - ek)^2 + (hd - eg)(kd - fg) \geq 0. \quad (2.15)$$

The monotonic behavior of r with k implies that if (2.12) holds for some $r = \xi_k^2$, then it must hold for all $r > \xi_k^2$. These results are summarized in the following theorem.

THEOREM 1. *For fixed even n :*

- (i) *All eigenvalues are real if (2.12) holds with $r = \xi_m^2$.*
- (ii) *All eigenvalues are complex if (2.12) does not hold for $r = \xi_1^2$.*
- (iii) *There are precisely $2p$ real eigenvalues if (2.12) holds for $r = \xi_k^2$, $k = 1, 2, \dots, p$ but not for $r = \xi_{p+1}^2$.*

For any even n :

- (iv) *All eigenvalues are real if (2.14) holds.*
- (v) *All eigenvalues are strictly complex if (2.15) does not hold.*
- (vi) *If (2.14) does not hold, there is an n (sufficiently large) such that at least some eigenvalues will become complex.*
- (vii) *If (2.15) holds, there is an n (sufficiently large) such that at least some eigenvalues will become real.*

In the case when B reduces to the identity matrix ($g=1$, $h=k=0$), the condition (2.12) is independent of r and reduces to

$$ef \geq 0$$

as the condition for real eigenvalues. Thus the eigenvalues are either all real or all complex and are given by the formula

$$\lambda = d + 2(ef)^{1/2} \cos \frac{k\pi}{n+1}, \quad k = 1, 2, \dots, n, \quad (2.16)$$

given by Mitchell [1].

Another interesting observation relating to the general case is that both $\text{Re}\lambda$ and $|\lambda|$ are monotonic functions of r . Thus when there are precisely $2p$ real eigenvalues [part (iii) of the above theorem], these eigenvalues are either those with smallest or those with largest real parts (moduli).

3. QUINDIAGONAL SYSTEMS

We now turn our attention to eigenvalue problems of the form (1.1) which arise from spatial discretization of parabolic differential equations by

quadratic finite elements (see [2]). Corresponding to (2.2), we define the $(2N+1) \times (2N+1)$ matrix

$$C = \begin{bmatrix} a & b & & & & & & \\ g & d & e & f & & & & \\ & c & a & b & & & & \\ & h & g & d & e & f & & \\ & & c & a & b & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & h & g & d & e \\ & & & & & & c & a \end{bmatrix} \quad (3.1)$$

and seek conditions under which $\det C = 0$. Premultiplying C by the matrix

$$Q = \begin{bmatrix} 1 & & & & & & & \\ \alpha & 1 & \beta & & & & & \\ & & 1 & & & & & \\ & & \alpha & \cdot & \cdot & & & \\ & & & & \alpha & 1 & \beta & \\ & & & & & & 1 & \end{bmatrix}$$

where $\alpha = -g/a$, $\beta = -e/a$, followed by a simple permutation of rows and columns, shows that $\det C$ vanishes when

$$\det \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} = 0, \quad (3.2)$$

where

$$C_{11} = \begin{bmatrix} d^* & f^* & & & \\ h^* & d^* & f^* & & \\ & & \cdot & \cdot & \\ & & & h^* & d^* \end{bmatrix}_{N \times N}, \quad C_{21} = \begin{bmatrix} b & & & & \\ c & b & & & \\ & c & b & & \\ & & \cdot & \cdot & \\ & & & c & b \\ & & & & c \end{bmatrix}_{(N+1) \times N},$$

$C_{22} = aI_{N+1}$ and

$$d^* = \frac{ad - bg - ec}{a}, \quad f^* = \frac{af - be}{a} \quad \text{and} \quad h^* = \frac{ah - gc}{a}.$$

Applying the results of the earlier lemma, we find

$$\det C = a \prod_{k=1}^N [(ad - bg - ec) - (af - be)^{1/2}(ah - gc)^{1/2}\xi_k], \quad (3.3)$$

where ξ_k is defined by (2.4). Setting $C = A - \lambda B$, the parameters a, b, \dots, h become linear functions of λ , and (3.3) gives

$$a = 0, \quad (3.4)$$

$$(ad - bg - ec)^2 = (af - be)(ah - gc)\xi_k^2, \quad 1 \leq k \leq \left[\frac{1}{2}N\right]. \quad (3.5)$$

In addition, when N is odd, there is the single quadratic equation

$$ad - bg - ec = 0, \quad k = \frac{1}{2}(N + 1). \quad (3.6)$$

Equations (3.4) to (3.6) provide respectively one linear, $[\frac{1}{2}N]$ quartic, and (when N is odd) one quadratic equation, from which the $2N + 1$ eigenvalues of (1.1) can be determined in the quindagonal case.

The eigenvectors of (1.1) in this case can also be determined by using (2.7) for the null eigenvectors of C_{11} in (3.2), and then relating these to the null eigenvectors of C through the transformation with Q . [When $a = 0$ the eigenvector can be determined directly from (3.1).]

4. FURTHER GENERALIZATIONS

The results of Secs. 2 and 3 can be extended in two directions, both having practical implications. In the first of these, we generalize (3.1) to matrices of the form

$$C = \begin{bmatrix} E & b & & & & \\ g^T & d & e^T & f & & \\ & c & E & b & & \\ & h & g^T & d & e^T & f \\ & & & \cdot & \cdot & \cdot \\ & & & h & g^T & d & e^T \\ & & & & & c & E \end{bmatrix}, \quad (4.1)$$

where E is a $p \times p$ matrix, $\mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{g}$ are $p \times 1$ vectors, and d, f, h are scalars. C is thus a square array of dimension $N + p(N + 1)$. This corresponds for example to using piecewise Lagrange polynomials of degree $p + 1$ in finite element applications. Following the approach adopted in Sec. 3, with α and β replaced by $\alpha^T = -g^T E^{-1}$ and $\beta^T = -e^T E^{-1}$, we find that

$$\det C = (\det E) \prod_{k=1}^N (d^* - (h^* f^*)^{1/2} \xi_k), \quad (4.2)$$

where $d^* = d(\det E) - g^T(\text{adj } E)\mathbf{b} - e^T(\text{adj } E)\mathbf{c}$, $f^* = f(\det E) - g^T(\text{adj } E)\mathbf{b}$, and $h^* = h(\det E) - e^T(\text{adj } E)\mathbf{c}$. The zeros of $\det C$ are therefore given by the p zeros of $\det E$, together with the roots of the polynomial equations

$$d^{*2} - h^* f^* \xi_k^2 = 0. \quad (4.3)$$

The degree of these equations (4.3) is $4p$ for $1 \leq k \leq \frac{1}{2}(N + 1)$, reducing to $2p$ when $k = \frac{1}{2}(N + 1)$, being generalizations of the quartic and quadratic equations [(3.5) and (3.6)] of the case $p = 1$ in the previous section. As in the previous section, there is no difficulty in principle to the determination of explicit expressions for the corresponding eigenvectors.

The second extension of Sec. 2 is to $nm \times nm$ matrices with m tridiagonal blocks, of the form

$$C = \begin{bmatrix} E & dI & & \\ eI & E & dI & \\ & \cdot & \cdot & \cdot \\ & & eI & E \end{bmatrix}, \quad (4.4)$$

where E is an $n \times n$ matrix of the form (2.2), I is the $n \times n$ unit matrix, and d, e are scalars ($de \neq 0$). This corresponds to the application of the finite element method to differential equations in two space dimensions.

Consider the partitioned vectors $\mathbf{u}^{(k,l)T} = (\mathbf{u}_1^{(k,l)T}, \dots, \mathbf{u}_m^{(k,l)T})$ where each subvector $\mathbf{u}_i^{(k,l)}$ has dimension n . Let

$$\mathbf{u}_i^{(k,l)} = \mathbf{v}^{(k)} w_i^{(l)}, \quad 1 \leq k \leq n, \quad 1 \leq l \leq n, \quad 1 \leq i \leq n,$$

where $\mathbf{v}^{(k)}$ is defined by (2.7) and

$$w_i^{(l)} = \left\{ -\left(\frac{e}{d}\right)^{1/2} \right\}^{i-1} \sin \frac{\pi i l}{m+1}, \quad i = 1, 2, \dots, m.$$

It is then straightforward although tedious, using elementary trigonometrical relationships, to show that if

$$a = \sqrt{ed} \eta_l + \sqrt{bc} \xi_k, \quad 1 \leq k \leq n, \quad 1 \leq l \leq n, \quad (4.5)$$

where $\eta_l = 2 \cos \{ \pi l / (m+1) \}$ and ξ_k is given by (2.4), then

$$C u^{(k,l)} = 0. \quad (4.6)$$

The vectors $u^{(k,l)}$ thus provide a full set of eigenvectors for the generalized eigenproblem (1.1) in this case, and the corresponding eigenvalues are given by (4.5). In the case when B is a unit matrix, (4.5) then provides an explicit expression for the eigenvectors. In general however the square roots must be removed from (4.5), and this necessitates squaring both sides of (4.5), rearranging, and squaring again, giving

$$(a^2 - ed\eta_l^2 - bc\xi_k^2)^2 = 4bc ed \xi_k^2 \eta_l^2. \quad (4.7)$$

Since a, b, \dots, e are linear functions of λ [arising as before from substituting given expressions for A and B in (1.1) into $C = A - \lambda B$ in (4.4)], it follows that (4.7) provides a quartic polynomial equation from which the eigenvalues of (1.1) can be determined in this case.

A further generalization to finite element methods in three space dimensions is also possible, although we omit the details. We only mention that an equation corresponding to (4.5) arises, but with three terms on the right hand side, and it requires squaring three times to remove the square roots. Thus the eigenvectors are given by the solution to a family of polynomial equations of degree eight.

5. APPLICATIONS

We have mentioned earlier that the motivation for analyzing problem (1.1) arose out of spatial discretizations of certain parabolic partial differential equations. Specifically, approximation of the equation

$$u_t = \epsilon u_{xx} - k u_x, \quad 0 < x < 1, \quad t > 0,$$

$$\text{subject to} \quad u(0, t) = u(1, t) = 0, \quad t > 0, \quad (5.1)$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1,$$

by finite element methods leads to a system of ordinary differential equations

$$B\dot{\mathbf{u}} = A\mathbf{u}, \quad (5.2)$$

where the dot denotes differentiation with respect to t . Solution of this system through Laplace transforms then leads to the system (1.1). The forms of the matrices A and B have been given in [2] for a variety of finite element schemes. The eigenvalues λ of the resulting system (1.1) correspond to approximations of the (real) decay rates of the Fourier components of the solution to (5.1). To some extent the presence of complex eigenvalues is an indication that the element size h is too large to provide reasonable accuracy in the approximation of the decay rates. This is discussed further in [2]. However, to illustrate the applications of our results, we shall summarize the findings for two typical situations.

EXAMPLE 1 (Piecewise linear finite elements). Defining the mesh length $H = 1/(n+1)$, the cell Peclet number $L = \frac{1}{2}KH/\epsilon$ and $\bar{\lambda} = \lambda H^2/\epsilon$, it can be shown that A and B are given by (2.1) with

$$d = -2, \quad e = 1 - L, \quad f = 1 + L,$$

$$g = 2H^2/3, \quad k = h = H^2/6.$$

Substituting these expressions into (2.10), we find that $\bar{\lambda}$ satisfies

$$(4 - \gamma_k^2)\bar{\lambda}^2 + 12(2 + \gamma_k^2)\bar{\lambda} + 36\{1 - (1 - L^2)\gamma_k^2\} = 0 \quad (5.3)$$

with $\gamma_k = \cos k\pi H$, $k = 1, 2, \dots, n$. Note that $\bar{\lambda}_{n+1-k} = \bar{\lambda}_k$. When n is odd, there is always a real root $\bar{\lambda}_{\frac{1}{2}(n+1)}$ at $\bar{\lambda} = -3$. The remaining roots will all be real if $L \leq \frac{3}{2}$ or all complex if $L^2 > 3$. For intermediate values of L the situation is governed by Theorem 1.

EXAMPLE 2 (Piecewise quadratic finite elements). In this case, $A - \lambda B$ is given by (3.1) where

$$a = 4\bar{\lambda}/5 + 8, \quad b = \bar{\lambda}/10 - 4 + 2L, \quad c = \bar{\lambda}/10 - 4 - 2L,$$

$$d = 4\bar{\lambda}/5 + 14, \quad e = \bar{\lambda}/5 - 8 + 4L, \quad f = -\bar{\lambda}/10 + 1 - L,$$

$$g = \bar{\lambda}/5 - 8 - 4L, \quad h = -\bar{\lambda}/10 + 1 + L.$$

There is always a real root $\bar{\lambda} = -10$ given by (3.4), and the remaining roots are obtained by substituting the above expressions into (3.5) and (3.6). That is

$$\begin{aligned} & \frac{1}{64}(9 - \gamma_k^2)\bar{\lambda}^4 + \frac{1}{4}(39 + 2\gamma_k^2)\bar{\lambda}^3 + \frac{1}{2}\{(383 + 15L^2) - (23 - 3L^2)\gamma_k^2\}\bar{\lambda}^2 \\ & + 20\{13(3 + L^2) + 2(3 - 2L^2)\lambda_k^2\}\bar{\lambda} + 100\{(3 + L^2)^2 - (9 - 3L^2 + L^4)\lambda_k^2\} = 0, \\ & 1 \leq k < \frac{1}{2}(n + 1), \end{aligned}$$

and

$$\frac{3}{8}\bar{\lambda}^2 + 13\bar{\lambda} + 10(3 + L^2) = 0$$

when $k = \frac{1}{2}(n + 1)$.

It can be shown that all eigenvalues are real provided $L \leq 2(\frac{31}{15})^{1/2}$.

Further analysis of the behavior of these eigenvalues, including their asymptotic behavior for large values of L , is detailed in [2].

When piecewise Lagrangian elements of degree $p + 2$ are applied to the solution of (5.1), the structure of the resulting matrix $A - \lambda B$ is given by (4.1). It is interesting to note from (4.2) that p of the eigenvalues can be determined from the analysis of a single element; $\det E = 0$ results from the eigenanalysis of an element in which the eigenfunctions vanish at the knots (end points of the element).

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Received 20 June 1978; revised 22 January 1979